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On Stationary Axially Symmetric Interior Solutions in General Relativity

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Abstract

This note presents the coordinate transformation by which the coordinate condition of a previous paper (Rawson–Harris, 1972) may be imposed.

Suppose that a metric is given in the coordinates $x^\mu = (z, r, \phi, t)$, in the region I of the previous paper, in the form

$$ds^2 = g_{11}(dz^2 + dr^2) + g_{33}\, d\phi^2 + 2g_{34}\, d\phi\, dt + g_{44}\, dt^2 \tag{1}$$

where $g_{33} = r^2 \gamma_{33}$, and also that

$$\underset{r\to 0}{\mathrm{Lt}}\ r^{-2} g_{33} g_{44} = \underset{r\to 0}{\mathrm{Lt}}\ \gamma_{33} g_{44} = -(h(z))^2 \tag{2}$$

for h non-zero. The differentiability of $g_{\alpha\beta}$, γ_{33}, and h will be considered later.

Any transformation of z, r to $\bar{z}, \bar{r}$ according to

$$D^2 \bar{r} \equiv \frac{\partial^2 \bar{r}}{\partial z^2} + \frac{\partial^2 \bar{r}}{\partial r^2} = 0 \tag{3a}$$

$$\frac{\partial \bar{z}}{\partial z} \equiv \bar{z}_{,z} = \bar{r}_{,r}, \qquad \bar{z}_{,r} = -\bar{r}_{,z} \tag{3b}$$

together with $\bar{\phi} = \phi$, $\bar{t} = t$, will preserve the form (1):

$$ds^2 = \bar{g}_{11}(d\bar{z}^2 + d\bar{r}^2) + \bar{g}_{33}\, d\bar{\phi}^2 + 2\bar{g}_{34}\, d\bar{\phi}\, d\bar{t} + \bar{g}_{44}\, d\bar{t}^2 \tag{4}$$

and also gives $\bar{g}_{33} = g_{33}$, $\bar{g}_{44} = g_{44}$. Let the metric $\bar{g}_{\alpha\beta}$ have the property that

$$\underset{\bar{r}\to 0}{\mathrm{Lt}}\ \bar{r}^{-2} \bar{g}_{33} \bar{g}_{44} = \underset{\bar{r}\to 0}{\mathrm{Lt}}\ \bar{\gamma}_{33} \bar{g}_{44} = -1 \tag{5}$$

Assume that the transformation may be such that

$$\bar{r} = rh(z) + O(r^{1+\delta}) \tag{6}$$

where $O(r^{1+\delta})$ refers to terms that are of order $r^{1+\delta}$ as $r \to 0$, $\delta > 0$, so that $\bar{r} \to 0$ as $r \to 0$. Then (5) gives

$$\begin{aligned}\operatorname*{Lt}_{\bar{r}\to 0} \bar{r}^{-2}\bar{g}_{33}\bar{g}_{44} &= \operatorname*{Lt}_{r\to 0} (r^2h^2 + O(r^{2+\delta}))^{-1} g_{33}g_{44} \\ &= \operatorname*{Lt}_{r\to 0} (r^2h^2 + O(r^{2+\delta}))^{-1} r^2 \gamma_{33} g_{44} \\ &= h^{-2} \operatorname*{Lt}_{r\to 0} \gamma_{33} g_{44} \\ &= -1\end{aligned}$$

i.e.

$$\operatorname*{Lt}_{r\to 0} \gamma_{33} g_{44} = -h^2$$

Thus (1) with (2) may be transformed into (4) with (5) and vice versa, if (3) and (6) may be satisfied for the function h.

To show that (3) and (6) may be satisfied, let $\bar{r} = rh(z) + u(z,r)$, so that (3a) implies that $D^2u = -rh_{,zz}$. If h is analytic, so that $g_{\alpha\beta}$ and γ_{33} are also, as is usual, then $u = r^3v(z,r) + C(z,r)$ where C are complementary functions: $D^2C = 0$. Thus $r = rh(z) + r^3v + C$. The conditions $C(z,0) = 0$, $C_{,r}(z,0) = 0$ are necessary for (6) and the transformation, and these, even applied on the part of the z-axis in I, are sufficient to set $C = 0$. Thus $\bar{r}$ is unique for given h, and $\bar{z}$ may be found from (3b) as it will always exist under the conditions assumed here.

If h, $g_{\alpha\beta}$, and γ_{33} are not analytic, but only C^3 at least, then either h must be continued in some way for the whole of the range $-\infty < z < \infty$, or boundary conditions additional to $\bar{r}(z,0) = 0, \bar{r}_{,r}(z,0) = h(z)$ must be imposed, say, on the remainder of the boundary of I or on lines $z =$ constant, $r > 0$ which form a semi-infinite strip containing I, in order that $\bar{r}$ be made unique for the given h. This problem is not a standard one because it involves Cauchy data for the Laplacian (see, e.g., Morse and Feshbach, 1953, Section 6.2) and one would have to consider individual cases.

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References

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